V.

PRELIMINARY WORK ON THE DETERMINATION OF THE LAW OF THE PROPAGATION OF HEAT IN THE INTERIOR OF SOLID BODIES.

By B. O. Peirce, Jr., and Edward B. Lefavour.

Presented Oct. 10, 1877.

For a long time it has seemed probable, as stated in a paper * published last spring in the Proceedings of the Academy, that the flux of heat in any direction x, in a solid body, can be written, —

$$-c\,\frac{df(v)}{dx}$$

where v is the temperature, c a constant different for different substances, and $f(v)$ an undetermined function of v. The object of this paper is to show that such a function $f(v)$ can be found.

This is not of necessity possible; for denoting by X, Y, Z, the components in three rectangular directions of the vector function which represents the flux, we have

$$-c\,\frac{dF(x, y, z)}{dx} = X$$

$$-c\,\frac{dF(x, y, z)}{dy} = Y$$

$$-c\,\frac{dF(x, y, z)}{dz} = Z$$

whence

$$-c\,dF(x, y, z) = X\,dx + Y\,dy + Z\,dz$$

This equation is integrable, and $F(x, y, z)$ can be found only when

$$\frac{dY}{dx} = \frac{dX}{dy} \qquad \frac{dZ}{dy} = \frac{dY}{dz} \qquad \frac{dX}{dz} = \frac{dZ}{dx}$$

* Note on the Determination of the Law of Propagation of Heat in Solid Bodies.

Our experiments were directed to determining whether these conditions can be satisfied, and

$$F(x, y, z) = f(v).$$

When a body heated in any way reaches a final state, — that is, a state where just the same quantity of heat enters each portion during a given time as leaves it, — the function $f(v)$, if it exists, must satisfv the equation

$$\frac{d_x^2 f(v)}{dx^2} + \frac{d_y^2 f(v)}{dy^2} + \frac{d_z^2 f(v)}{dz^2} = 0.$$

One, and only one, solution of this equation corresponds to each set of physical conditions. Since v, and any function of v, are constant along the same surfaces, if, when the body is in a final state, v is constant along each surface of the family $\varphi\ (x, y, z) = k$, where $\varphi\ (x, y, z)$ is the solution of Laplace's equation, corresponding to the given physical conditions, then it is always possible to find a function of the temperature alone, which shall satisfy Laplace's equation as a function of x, y, z, or, what is the same thing, shall be equal to $\varphi\ (x, y, z)$ throughout all space.

For let u and v be two functions of x, y, z, such that $u = k$, and $v = c$ represent the same family of surfaces, then denoting by du the total differential of u, and by $d_x u$ the partial differential relative to x,

$$\frac{\frac{d_x u}{dx}}{\frac{d_y u}{dy}} = \frac{\frac{d_x v}{dx}}{\frac{d_y v}{dy}}, \qquad \frac{\frac{d_y u}{dy}}{\frac{d_z u}{dz}} = \frac{\frac{d_y v}{dy}}{\frac{d_z v}{dz}} \qquad \frac{\frac{d_z u}{dz}}{\frac{d_x u}{dx}} = \frac{\frac{d_z v}{dz}}{\frac{d_x v}{dx}},$$

$$\therefore \quad \frac{\frac{d_x u}{dx}}{\frac{d_x v}{dx}} = \frac{\frac{d_y u}{dy}}{\frac{d_y v}{dy}} = \frac{\frac{d_z u}{dz}}{\frac{d_z v}{dz}} = \psi\ (x, y, z).$$

If p is any variable,

$$\frac{d_p u}{dp} = \frac{d_x u}{dx} \cdot \frac{d_p x}{dp} + \frac{d_y u}{dy} \cdot \frac{d_p y}{dp} + \frac{d_z u}{dz} \cdot \frac{d_p z}{dp}$$

And

$$\frac{d_p v}{dp} = \frac{d_x v}{dx} \cdot \frac{d_p x}{dp} + \frac{d_y v}{dy} \cdot \frac{d_p y}{dp} + \frac{d_z v}{dz} \cdot \frac{d_p z}{dp}$$

or, substituting,

$$\frac{\frac{d_p u}{dp}}{\frac{d_p v}{dv}} = \psi\ (x, y, z).$$

Similarly the ratio of the corresponding total differential coefficients is

$$\frac{\frac{du}{dp}}{\frac{dv}{dp}} = \psi\ (x, y, z).$$

Whence, changing the variable and integrating,

$$\frac{d_v u}{dv} = \psi\ (x, y, z) = \frac{du}{dv}$$

The partial and total differential coefficients of u taken relatively to v cannot be equal, if u involve any other quantity than v,

$$\therefore\quad u = f(v)$$

In short, when a body is heated in any manner whatever, there must exist a function $f(v)$, the same for all bodies, whose derivative in any direction, when multiplied by a constant depending on the nature of the body, gives the flux of heat in that direction, provided v is found constant along the surfaces $\varphi\ (x, y, z) = k$, which belong to the solution of Laplace's equation for that particular case.

The first case open to direct and satisfactory experiment is where a plate of metal is heated at two points, and exposed to the air only at its edges. The isothermals in this case belong to the family $A \log r_1 + B \log r_2 = k$;

or

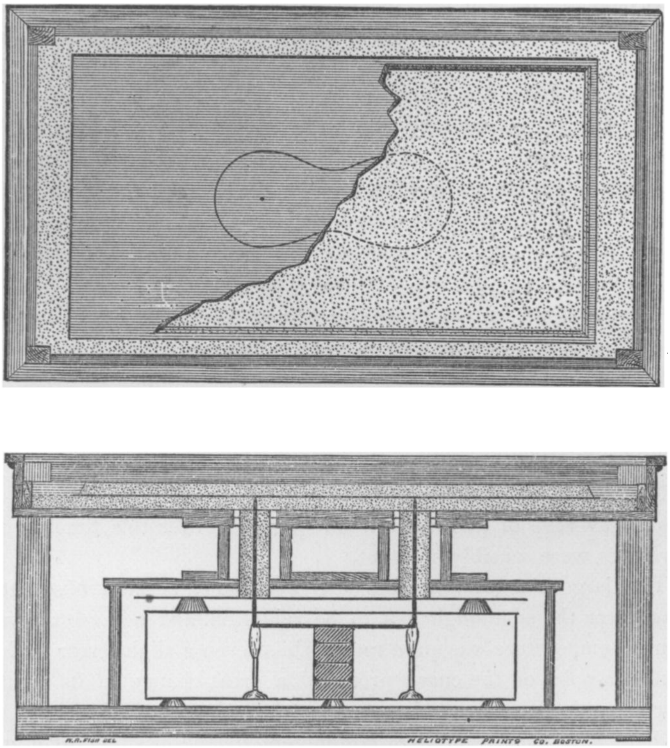
$$r_1 r_2^{\,m} = c.$$

This latter form of the equation shows that the two points need not be *heated equally.* The solution for three dimensions cannot be readily submitted to experiment; but the probability that $f(v)$ should satisfy the solution for two dimensions, and fail in that for three, is so slight that it may be neglected.

Our first experiments were with a small iron plate covered with a mixture of wax, rosin, and paraffine. By this means, one curve was obtained at each heating; viz., that separating the part of the mixture which had melted from that which remained solid. This method can

be used in the open air, but is impracticable when the waxed plate is covered by a non-conducting material.

After some further rude work, we constructed the table of which a diagram is given below: —



Two Bunsen burners are enclosed in an iron case, which is itself enclosed in a wooden case, surrounding it at a distance of 5 to 7 cm. Through the top of the iron case is put a bent copper rod, heated at either elbow by one of the Bunsen flames. The wooden case rests on a floor fastened beneath the table. The ends of the rod, rising from the iron case, and surrounded and held firmly in place by tin cylinders closed at both ends, and themselves secured to enclosing wooden pipes, project some 5 cm. above the top of the table.

Around this top is a guard, within which are wooden cleats, by means of which to level the non-conducting material. This non-con-

ductor is the almost pure silicious earth, dug in Keene, N. H., and known as "infusorial earth." With it, we were abundantly supplied by the kindness of Messrs. J. A. Wright and James H. Wilson of Keene. It is in every way suited to work of this kind, being clean, free from any appreciable amount of moisture, and an almost perfect non-conductor of heat.

Upon the ends of the copper rod, covered above and below to within 2 or 3 cm. of the edges, rests a sheet of No. 11 boiler-plate iron, 1.5 meters long and .9 meters wide.

The first thing necessary to the success of our experiments with the plate was that the head of gas should be constant. We found the variation to be insensible, although a change in pressure of a fraction of one mm. of water could have been detected.

The temperature of the air must also be constant. To secure this, we were obliged to use the precautions mentioned above. The air for the lamps entered at the bottom of the cases, and passed out from both by an iron pipe placed within a wooden one. The tin cylinders were filled with "infusorial." The variation in the temperature of the air was less than one degree during the day. It would have been impossible to keep the temperature exact to the $\frac{1}{1000}$ of a degree, as is reported of Biot.

No work could be done until the plate reached a final state. For this an average of five hours was required. After this time the temperatures were sensibly constant.

The large size of the plate was necessary, because observations taken near the edge ought not to be relied on, and because the variation in temperature was quite rapid, whilst even a slight error in determining a point on the curve produced a great change in its equation. For example, in one case, in a curve whose loop was about 210 mm. in diameter, a change of *four* millimeters in the position of a given point changed m in the equation $r_1 r_2^m = c$ from $-.5$ to $+.8$, or the equation from

$$\frac{r_1}{r_2^{\frac{1}{2}}} = c \text{ to } r_1 r_2^{\frac{4}{5}} = c.$$

In another case, a change of three millimeters altered the equation from

$$r_1 r_2^{8} = c \text{ to } r_1 r_2^{\frac{1}{2}} = c$$

The measurements were made with the galvanometer and thermopile. Our galvanometer has four coils, whose resistance is 11. ohms, and a mirror of 1 meter focal length. Each scale-division is $\frac{5}{8}$ mm.

In order that the electromotive force of the current, and therefore the deflections of the galvanometer, should be directly proportional to the excess of temperature of the point touched over that of the air, a pair of metals must be taken whose neutral point is at a very high or very low temperature, and whose relation to each other varies the least possible with the temperature. Iron and German-silver satisfy these conditions remarkably well. Their neutral point is at 1354°, as deduced from Prof. Tait's table. Some rude experiments of ours on the relation between the temperature and the deflections gave quite ragged curves; but they were evidently in every case straight lines. The three lines obtained by heating and cooling water were nearly parallel. The mean of these experiments showed that one degree of temperature corresponds to 4.3 scale-divisions. One microvolt corresponds to 2.6 scale-divisions, or one scale-division to .39 microvolts. We thus find that one degree of change in temperature produces an electromotive force of 18.3 microvolts. The value as calculated by Prof. Tait's formula is 21.8.

Our first experiments were with two thermopiles. The method of work was this. Each thermopile consisted of one joint, which was fitted to a piston moving vertically 3 cm. in a brass cylinder. The piston was depressed by a weight resting on a lever, whose fulcrum was on the frame around the table, and its end at the thermopile. The average pressure applied was 9 kilos. This was opposed by the reaction of springs, which raised the piston when the weight was removed. The cylinders were placed on carriages, running on wooden strips, which crossed the table transversely just above the infusorial. These strips and the sides of the table were graduated, and thus we got the (x, y) co-ordinates of the points touched.

One thermopile was kept in a fixed position, the other was moved about to various points where equal deflections were obtained. All these last points lie on an isothermal. Great care is necessary that the contact with the plate be good, and that the piston move in an exactly vertical direction. The results thus found were unsatisfactory, because, as before pointed out, an inappreciable error in observation produces a very perceptible change in the equation of the curve. Another objection is that the method requires a great expenditure of time.

A second method is more satisfactory. One thermopile was moved lengthwise along the plate, and the deflections laid off in a curve. The same may also be done transversely, but the curves obtained are of little value.

The results of our work on "longitudinal curves" are given below. Evidently, in crossing the plate, we shall find, in general, four points having the same temperature, two about either pole. The isothermal points lying about one pole in each of two experiments, obtained by averaging several readings of the curves mapped out, are as follows: —

ISOTHERMAL POINTS.

Exp. 1.	Exp. 2.	Mean.
488 — 658.1	488 — 661.9	488 — 660.
495 — 644.4	495 — 643.4	495 — 643.9
500 — 632.6	500 — 631.1	500 — 631.85
502½ — 626.8	502½ — 625.9	502½ — 626.35
505 — 622.5	505 — 620.6	505 — 621.55
507½ — 617.4	507½ — 615.9	507½ — 616.65
510 — 612.9	510 — 611.8	510 — 612.35
515 — 604.4	515 — 602.3	515 — 603.85
517 — 600.9	517 — 599.4	517 — 600.15
520 — 596.9	520 — 595.6	520 — 596.25
525 — 591.3	525 — 588.8	525 — 590.05
527 — 588.4	527 — 586.6	527 — 587.5
530 — 584.6	530 — 582.5	530 — 583.55
535 — 578.7	535 — 576.2	535 — 577.45
555 — 555	555 — 555	555 — 555

The isothermal curves corresponding to these pairs of points are in general form like the family to which the common lemniscate belongs. Several other curves have much the same form, and therefore it becomes necessary to show that our experimental curve is none of these. The following conditions must be satisfied: —

(1) For a given positive value of r_1 there must be only one real and positive value of r_2, r_1 and r_2 being radii from the fixed points.

(2) The function must be homogeneous; else, for the same physical problem we shall get as many different solutions as we employ units of measure.

Hence we must exclude trigonometric functions, by reason of their periodicity, and, since exponential and logarithmic forms may be developed in algebraic series, need consider only these last.

In this system of coördinates, negative, as well as truly imaginary values of r_1 and r_2, are to be regarded as imaginary. (1) requires us to

exclude any form in which both positive and negative terms enter, as also forms where the exponents of the powers of r are both positive and negative.

Three simple equations satisfy these conditions: —

$$(1.)\ r_1 r_2 = c. \quad (2.)\ \frac{1}{r_1} + \frac{1}{r_2} = c. \quad (3.)\ r_1 + r_2 = c.$$

(3) represents the ellipse; and all forms $r_1^m + r_2^m = c$ are of equally little use to us.

The isothermal points for $\frac{1}{r_1} + \frac{1}{r_2} = c$ are approximately: —

488 — 627	515 — 595½
495 — 620	517 — 593
500 — 615½	520 — 590
502½ — 612	525 — 585½
505 — 608½	527 — 583½
507½ — 605	530 — 580
510 — 602½	535 — 575

The isothermal curves for $\frac{1}{r_1^2} + \frac{1}{r_2^2} = c$ are much nearer circles, and the isothermal points more nearly equidistant from the pole. This approximation to a series of circles increases with the degree of the equation. There remains to be considered only $\Sigma\, r_1^{m+x} r_2^{m-x} = c$.

The physical conditions require that the two loops of the isothermal curves shall be approximately equal. But, whenever the ratio $\frac{m+x}{m-x}$ in the equation $r_1^{m+x} r_2^{m-x} = c$ differs much from 1, the loops are very unequal; and any term in which a ratio of this sort is found introduces into the equation $\Sigma\, r_1^{m+x} r_2^{m-x} = c$ an inequality unbalanced by the other terms.

Assuming now $r_1 r_2^p = c$, as the equation of our experimental curve, the values of p, corresponding to the isothermal points in the "Mean" column, are as follows: —

(1)	1.097	(8)	.988	
(2)	1.125	(9)	.912	
(3)	1.093	(10)	.950	
(4)	1.072	(11)	1.056	average 1.059
(5)	1.065	(12)	1.089	
(6)	1.041	(13)	1.098	
(7)	1.035	(14)	1.207	

Compare now the experimental values before given with the four series of values below.

The third and fourth fail to satisfy condition (2), as this condition was discovered too late to change our figures, but they will serve equally well for illustration.

Values corresponding to	in $r_1 r_2^{1.05} = c$*	in $r_1 r_2^{1.075} = c$*	in $r_1 r_2^{.9} + r_1 r_2^{1.2} = c$†	in $r_1 r_2^{.9} + r_1 r_2^{1.25} = c$†
488	657.1	658.5	662.	668.
495	640.8	641.8	643.	649.
500	630.6	631.3	632.	637.
502½	625.8	626.4	625.	630.5
505	621.2	621.8	620.5	625.
507½	616.8	617.3	616.5	620.
510	612.6	613.	612.	615.
515	604.6	604.8	602.	606.5
517	601.5	601.8	600.	602.5
520	597.1	597.3	595.	598.
525	590.	590.2	588.	591.5
527	587.3	587.4	585.5	588.
530	583.4	583.5	581.	583.
535	577.1	577.15	575.5	577.

It will appear at once that while the first two curves lie very close to the experimental curve, and parallel with it, even such close approximations to the one-term equation as those above cut at an angle. The use of residual curves will make the fact still clearer. Equations of three, four, or more terms, will yield like results.

Even if an equation could be found of this kind which was equally good with the simple form, it would still be improbable that the simpler should not be the form preferred by nature.

Since then the temperature is shown experimentally to be constant along the family of curves $r_1^m r_2 = c$, a function $f(v)$ can be found, the same for all bodies, whose derivative in any direction in a given body, when multiplied by a constant quantity depending on the nature of the body, measures the flux of heat in that direction.

* Computed. † Approximate.

The solution for one dimension $\left[\frac{d_x^2 f(v)}{dx^2} = 0\right]$, as well as that for two, admits of experimental verification.

Let a long rod, covered with non-conducting material, except at the ends, be heated at the points A and B. Then for A: $f(v_1) = k_1 r_1 + b_1$. For B: $f(v_2) = k_2 r_2 + b_2$; r_1 and r_2 being the distances from A and B measured on the rod.

$$f(v) = f(v_1) + f(v_2) = k_1 r_1 + k_2 r_2 + b_1 + b_2 = k_1 r_1 + k_2 r_2 + \lambda.$$

$$\text{Let } k_1 = k_2 \qquad fv = k_1(r_1 + r_2) + \lambda.$$

$f(v)$ between A and B is constant, and may be represented by a straight horizontal line. Whenever $f(v)$ has a constant value, v has a constant value, and conversely. Therefore, v, as well as $f(v)$, must be represented between A and B by a straight line. If A and B are not heated exactly alike, the line will still be straight, but inclined to the horizontal.

We have not yet tried the experiment, but intend to do so, that there may be the greatest possible assurance in regard to the existence of $f(v)$, seeing that it exists in every case which can be experimentally tested.

The second portion of our work is the determination of $f(v)$. For a rod heated to the final state, it must satisfy the condition $\frac{d_x^2 f(v)}{dx^2} = 0$, or $f(v) = Ax + B$.

For a plate: $\frac{d_x^2 f(v)}{dx^2} + \frac{d_y^2 f(v)}{dy^2} = 0$, or $f(v) = C + D \log r$ (changing to polar co-ordinates).

For a solid: $\frac{d_x^2 f(v)}{dx^2} + \frac{d_y^2 f(v)}{dy^2} + \frac{d_z^2 f(v)}{dz^2} = 0$, or $f(v) = E + \frac{F}{r}$.

This function cannot be v itself. For, in the case of a rod, heated at one point, let $fv = v$.

$$\begin{array}{lll} & v = Ar + B. & \\ \text{When } r = 1 & v = A + B \mp \infty & \left\{ \begin{array}{l} \text{since by physical conditions} \\ \text{the temperature must here} \\ \text{be finite.} \end{array} \right. \\ \quad ,, \quad r = 0 & v = B \mp \infty & \\ \quad ,, \quad r = \infty & v = A\,\infty + B = 0, & \text{which is impossible.} \end{array}$$

For a plate

$$f(v) = v = C + D \log r$$

$$\begin{array}{ll} \text{When } r = 1 & v = A \mp \infty \\ \quad ,, \quad r = 0 & v = C + (-\infty), \end{array}$$ which again is contrary to the physical conditions.

For a solid

$$f(v) = v = E + \frac{F}{r}$$

When $r = \infty$ $\quad v = 0 \quad \therefore E = 0 \quad \therefore v = \frac{F}{r}$

" $\quad r = 0 \quad v = \frac{F}{r} = \infty$, which again is impossible.

The same thing may be proved as follows. Assume, for the sake of argument, that the flux is $-c\frac{d_x v}{dx}$ for the direction x, $-c\frac{d_y v}{dy}$ for y, $-c\frac{d_z v}{dz}$ for z. Then for all homogeneous bodies,

$$\frac{dv}{dt} = \lambda\left(\frac{d_x{}^2 v}{dx^2} + \frac{d_y{}^2 v}{dy^2} + \frac{d_z{}^2 v}{dz^2}\right).$$

This is the only condition that a function v must satisfy, in order to represent an actual possible case. Any function that is a particular solution of this equation may represent an actual distribution of heat.

First. If $v = \varphi(x, y, z, t)$ represent the temperature throughout a body, it cannot have any true maxima and minima for x, y, z. This is evident, since the conditions of a maximum or minimum are $\frac{d_x v}{dx} = 0$, $\frac{d_y v}{dy} = 0$, $\frac{d_z v}{dz} = 0$, which by hypothesis cannot be zero without making the flux $= 0$. No point can be hotter or colder than the points around it, and there not be a flux to or from the point. The physical conditions forbid the mathematical condition of the existence of maxima and minima. There must be points hotter than points around them, and therefore they must be shooting points, and not maxima or minima.

Secondly. If v_1 and v_2 are two particular solutions of the Partial Differential Equation, their sum is also a solution, and therefore corresponds to an actual distribution of heat, in which the temperature of any point is equal to the sum of the temperatures, which it would have under the conditions represented by v_1 and v_2. It will now be easy to show

Thirdly. The points of hottest temperature, when the solution is v, fall in exactly the same places in the body as the hottest points of the two solutions v_1 and v_2 taken jointly. That is to say: if there are n_1 points of highest temperature when the solution is v_1, and n_2 when the solution is v_2, there will be $n_1 + n_2$ points of highest temperature when the solution is $v_1 + v_2 = v$, unless v_1 and v_2 have some hot points in

common, or else a minimum of v_1 or v_2 corresponds with a maximum of v_2 or v_1. Here special investigation is necessary for each particular case.

Observe carefully that, if the functions v, v_1 and v_2 attained *true* maxima or minima at the hot points, instead of coming to a point, the proposition would not be true.

That the proposition is true, in regard to shooting points, can be seen thus: $\frac{d_x v}{dx} = \tan \tau$; $\tan \tau$ changes sign instantaneously at a shooting point, and hence, for a small (infinitely small) change in $x[dx]$, $\frac{d_x v}{dx}$ has a finite change of value, and $d_x \frac{d_x v}{dx} =$ finite, $dx = \frac{1}{\infty}$. Therefore

$$\frac{d_x \frac{d_x v}{dx}}{dx} = \frac{\textit{finite}}{\frac{1}{\infty}} = \infty$$

This is the condition for a shooting point. If, at the same point, $\frac{d_x v}{dx}$ as well as $\frac{d_x^2 v}{dx^2}$ is infinite, there is a cusp. If also $v = \infty$, there is an asymptote.

At the points of highest temperature $\frac{d_x^2 v}{dx^2} = \infty$, $\frac{d_y^2 v}{dy^2} = \infty$, $\frac{d_z^2 v}{dz^2} = \infty$.

$$\text{Now} \quad \frac{d_x^2 v}{dx^2} = \frac{d_x^2 v_1}{dx^2} + \frac{d_x^2 v_2}{dx^2}$$

and any values of x, y, z, which make either term in the second member infinite, will make the whole member infinite, and insure a shooting point for v as far as the direction x is concerned, wherever there is one for either v_1 or v_2. The like is true of y and z. If $\frac{d_x^2 v_1}{dx^2}$ and $\frac{d_x^2 v_2}{dx^2}$ become ∞ with opposite signs, there will be an ambiguity which can be got rid of by the determination of the indeterminate quantity ($\infty - \infty$).

Using the results of both (2) and (3), it is easy to arrive at an absurdity in almost any case considered. For instance, heat to a constant temperature all the points upon a circle, marked out upon a large metal plate protected from surface radiation. Then not only will the point at the centre of the circle soon become hotter than any point upon the rim, in which case the heat must flow from within out, but it must finally be infinitely hot, which is absurd.

Our experiments have not proceeded so far that we can determine

$f(v)$ from them. That determination must form the subject of a subsequent paper. We will simply present here a specimen of our experimental results for both the rod and the plate, heated at one point under the conditions above specified; the values in column "v" being galvanometer deflections: —

Rod. r	Rod. v	Plate. r	Plate. v
0	342	0	178
100	257	50	108
200	196	100	83
300	150	150	61½
400	111	200	53
500	87	300	37
600	67	400	28½
700	54	500	20½
800	43	600	15
900	33		
1000	29		

From these series we are to determine v as a function of r. If $v = \varphi(r)$. Then for a rod, $f\varphi(r) = Ar + B$; for a plate, $f\varphi(r) = C + D \log r$.

Permit us here to express our great gratitude to your Academy for the generosity and liberality which have supplied us with the means for carrying on our experiments.

Harvard University, Oct. 10, 1877.